PROBLEM OF CONTROL OF THE LEVEL OF GROUND-WATER DURING IRRIGATION FOR THE THREE-DIMENSIONAL CASE

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In [1], the problem concerning the control of the level of groundwater during irrigation is considered. It is solved on the assumption that the surface of the ground flow is weakly curved, the watertight layer is weakly permeable, is level, and has a constant thickness M_0, and that the groundwaters occupy the region between two parallel channels or drains.

It will be interesting to solve the similar problem in the two-dimensional region between four channels forming the rectangle $0 \le x \le l$, $0 \le y \le L$. This solution is obtained in this paper in explicit form. In this case, the following method of controlling the level of groundwater can be represented as follows: irrigation, producible with intensity me/θ, ceases when the level of the groundwater, measured at a fixed point $0 \le x^0 \le l$, $0 \le y^0 \le L$ of the region between the channels, reaches a quantity h_*, and begins again when $h(x^0, t)$ becomes equal to $h_{**} < h_*$ $(0 < \theta < 1)$. This problem reduces to solving the equation

$$\frac{\partial h}{\partial t} = a^2\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right) - b(h - H) + F[h(x^0, y^0, t)], \tag{1.1}$$

where

$$F[h(x^0, y^0, t)] = \begin{cases} c & \text{for} \quad h(x^0, y^0, t) < h_* \\ -d & \text{for} \quad h(x^0, y^0, t) > h_{**} \end{cases} \tag{1.2}$$

(the notation is the same as in [1]; θ is the ratio of the intensities of filtration and wetting), with a certain initial condition and boundary conditions (Fig. 1)

$$h(x, 0, t) = h(x, L, t) = H_1 + (H_2 - H_1)\cdot x/l, \qquad h(0, y, t) = H_1, \; h(l, y, t) = H_2. \tag{1.3}$$

We put

$$h(x, y, t) = H_1 + (H_2 - H_1)\cdot x/l + \tilde{u}(x, y, t). \tag{1.4}$$

For $\tilde{u}(x, y, t)$, we obtain the equation

$$\frac{\partial \tilde{u}}{\partial t} = a^2\left(\frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{u}}{\partial y^2}\right) - b\left[H_1 - H + (H_2 - H_1)\frac{x}{l}\right] - b\tilde{u} + F(\tilde{u}), \tag{1.5}$$

where $\tilde{F}(u)$ is given by formulas (1.2) and (1.4) with the conditions

$$\tilde{u}(x, 0, t) = \tilde{u}(x, L, t) = \tilde{u}(0, y, t) = \tilde{u}(l, y, t) = 0. \tag{1.6}$$

It can be seen that Eq. (1.5) with conditions (1.6) has stable stationary solutions

$$v(x, y) = \sum_{k=1}^{\infty} \frac{\mu_k^{(1)}}{\operatorname{sh} \lambda_k L}[\operatorname{sh} \lambda_k (y - L) - \operatorname{sh} \lambda_k y + \operatorname{sh} \lambda_k L] \sin\frac{\pi k x}{l},$$

$$w(x, y) = \sum_{k=1}^{\infty} \frac{\mu_k^{(2)}}{\operatorname{sh} \lambda_k L}[\operatorname{sh} \lambda_k (y - L) + \operatorname{sh} \lambda_k y + \operatorname{sh} \lambda_k L] \sin\frac{\pi k x}{l},$$

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$$\lambda_k^2 = \frac{\pi^2 k^2}{l^2} + \frac{b}{a^2}, \quad \mu_k^{(i)} = \frac{2\{[-(-1)^k + 1][c_i - b(H_1 - H)] + b(H_2 - H_1)(-1)^k\}}{\pi a^2 \lambda_k^2 k}, \tag{1.7}$$

$$c_1 = c, \quad c_2 = -d.$$

The periodic solution of Eq. (1.5) with the conditions (1.6) has the form

$$\widetilde{u}_1(x, y, t) = v(x, y) + \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} C_{k,m} \exp(-\lambda_{k,m}^2 t) \sin\frac{\pi k x}{l} \cdot \sin\frac{\pi m y}{L} \quad (0 \leqslant t \leqslant T_1),$$

$$\widetilde{u}_2(x, y, t) = w(x, y) + \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} D_{k,m} \exp[-\lambda_{k,m}^2 (t - T_1)] \sin\frac{\pi k x}{l} \cdot \sin\frac{\pi m y}{L} \quad (T_1 \leqslant t \leqslant T), \tag{1.8}$$

where we introduce the notation

$$C_{k,m} = \frac{-\theta_{k,m}(1 - \gamma_{k,m})}{1 - \delta_{k,m}}; \quad D_{k,m} = \frac{\theta_{k,m}(1 - \beta_{k,m})}{1 - \delta_{k,m}}; \tag{1.9}$$

$$\lambda_{k,m}^2 = \pi^2 a^2 \left(\frac{k^2}{l^2} + \frac{m^2}{L^2}\right) + b;$$

$$\beta_{k,m} = \exp(-\lambda_{k,m}^2 T_1); \quad \gamma_{k,m} = \exp[-\lambda_{k,m}^2 (T - T_1)], \quad \delta_{k,m} = \beta_{k,m}\gamma_{k,m},$$

$$\theta_{k,m} = -\frac{4(c+d)}{\pi a^2 L} \frac{[(-1)^k - 1][(-1)^m - 1]}{k\lambda_k^2} \left\{\frac{\pi m}{L\left(\lambda_k^2 + \frac{\pi^2 m^2}{L^2}\right)} - \frac{L}{\pi m}\right\}.$$

Here T is the period of self-oscillation; T_1 is the duration of the wetting stage. The constants T_1 and $T_2 = T - T_1$ are defined as the smallest roots of the system of two equations

$$\widetilde{u}_1(x^0, y^0, T_1) = u_*, \quad \widetilde{u}_2(x^0, y^0, T) = u_{**}. \tag{1.10}$$

In view of Eq. (1.8), Eq. (1.10) reduces to the form

$$\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} C_{k,m}\beta_{k,m} \sin\frac{\pi k x^0}{l} \cdot \sin\frac{\pi m y^0}{L} = U_*, \tag{1.11}$$

$$\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} D_{k,m}\gamma_{k,m} \sin\frac{\pi k x^0}{l} \cdot \sin\frac{\pi m y^0}{L} = U_{**}.$$

Here, for brevity, we introduce the notation

$$U_* = u_* - v(x^0, y^0), \quad U_{**} = u_{**} - w(x^0, y^0). \tag{1.12}$$

Using Eqs. (1.12) and (1.9), we rewrite formula (1.11) in the following form:

$$\widetilde{\varphi}(T_1, T_2) = U_*, \quad \widetilde{\psi}(T_1, T_2) = U_{**}.$$

Here we denote

$$\widetilde{\varphi}(T_1, T_2) = -\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{\theta_{k,m}\beta_{k,m}(1 - \gamma_{k,m})}{1 - \delta_{k,m}} \sin\frac{\pi k x^0}{l} \sin\frac{\pi m y^0}{L}, \tag{1.13}$$

$$\widetilde{\psi}(T_1, T_2) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{\theta_{k,m}\gamma_{k,m}(1 - \beta_{k,m})}{1 - \delta_{k,m}} \sin\frac{\pi k x^0}{l} \cdot \sin\frac{\pi m y^0}{L}.$$

Expanding the double series $S = \sum_{m,k=1}^{\infty} a_m^{(k)}$ in the form of an infinite rectangular matrix, we then represent its terms in the form of a simple sequence as far as the squares:

$$S = \sum_{p=1}^{\infty} a_p, \quad \text{whereby} \quad a_m^{(k)} = a_p.$$

Here $P = (k-1)^2 + m$ when $m \leq k$ and $p = m^2 + 1 - k$ when $m > k$. We represent the double series in formulas (1.13) in the form of regular series

$$\widetilde{\varphi}(T_1, T_2) = \sum_{p=1}^{\infty} a_p(T_1, T_2), \quad \widetilde{\psi}(T_1, T_2) = \sum_{p=1}^{\infty} bp(T_1, T_2),$$

where

$$a_p = \frac{-\theta_{k,m}\beta_{k,m}(1-\gamma_{k,m})}{1-\delta_{k,m}} \sin\frac{\pi k x^0}{l} \cdot \sin\frac{\pi m y^0}{L};$$

$$b_p = \frac{\theta_{k,m}\gamma_{k,m}(1-\beta_{k,m})}{1-\delta_{k,m}} \sin\frac{\pi k x^0}{l} \cdot \sin\frac{\pi m y^0}{L}.$$

In formulas (1.13) we put $T = \nu T_1 (\nu \geq 1)$, we obtain $\tilde{\varphi}(0, 0) = [(\nu-1)/\nu]\theta(x^0, y^0); \tilde{\psi}(0, 0) = (1/\nu)\,\theta(x^0, y^0)$, $(\theta(x^0, y^0) = v(x, y) - w(x, y))$. Graphs of the dependences

$$Y = \tilde{\varphi}a^2/(c+d)l^2 < 0, \; Z = \tilde{\psi}a^2/(c+d)l^2 > 0$$

on $T_1' = a^2 T_1/l^2$ are shown in Fig. 2, where curves 1-3 correspond to the values $\nu = 4/3$, $\nu = 2$, and $\nu = 4$, respectively. The solid curves correspond to the value $b' = 0$, and the dashed curves correspond to $b' = 1/60$ ($b' = bl^2/a^2$). It is clear from the form of these curves that to a pair of values of $\tilde{\varphi}$ and $\tilde{\psi}$, such that $-\theta(x^0, y^0) < \tilde{\varphi} < 0$, $0 < \tilde{\psi} < \theta(x^0, y^0)$, and $\tilde{\psi} - \tilde{\varphi} < \theta(x^0, y^0)$, there corresponds in an unambiguous way a pair of values $T_1 > 0$ and ν (or values $T_1 > 0$ and $T_2 > 0$). Consequently, to each pair of values u_* and u_{**}, such that $w(x^0, y^0) < u_{**} < u_* < v(x^0, y^0)$, there corresponds a single pair of values $T_1 > 0$ and $T_2 > 0$.

§2. We shall consider the solution of the initial value problem (1.1) and (1.2) with the conditions (1.3) and the starting condition

$$h(x, y, 0) = \varphi(x, y), \tag{2.1}$$

or (which is the same), the problem (1.5) and (1.2) with the conditions (1.6) and

$$\tilde{u}(x, y, 0) = \psi(x, y) \quad (\psi(x, y) = \varphi(x, y) - H_1 - (H_2 - H_1)\cdot x/l). \tag{2.2}$$

If the solution of this problem exists, then it has a form similar to Eq. (1.8):

$$\tilde{u}_1^{(i+1)}(x, y, t) = v(x, y) + \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} C_{k,m}^{(i+1)} \exp\left[-\lambda_{k,m}^2 (t - T'^{(i)})\right] \sin\frac{\pi k x}{l} \cdot \sin\frac{\pi m y}{L} \tag{2.3}$$

$$\Big(T'^{(i)} \leqslant t \leqslant T_1'^{(i+1)},\; T'^{(i)} = \sum_{j=0}^{i} T^{(j)},\; T_1'^{(i+1)} = T'^{(i)} + T_1^{(i+1)},$$

$$i = 0, 1, 2, 3, \ldots, T^{(0)} = 0\Big),$$

$$\tilde{u}_2^{(i+1)}(x, y, t) = w(x, y) + \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} D_{k,m}^{(i+1)} \exp\left[-\lambda_{k,m}^2 (t - T_1'^{(i+1)})\right] \sin\frac{\pi k x}{l} \cdot \sin\frac{\pi m y}{L}$$

$$(T_1'^{(i+1)} \leqslant t \leqslant T'^{(i+1)}).$$

Here $C_{k,m}^{(1)}$ are the coefficients of the Fourier function $\psi(x, y) - v(x, y)$:

$$C_{k,m}^{(1)} = \frac{4}{lL}\int_0^l\int_0^L [\psi(x, y) - v(x, y)] \sin\frac{\pi k x}{l} \cdot \sin\frac{\pi m y}{L}\, dx dy; \tag{2.4}$$

$C_{k,m}^{(i+1)}$ $(i = 0, 1, 2, \ldots)$ and $D_{k,m}^{(i+1)}$ $(i = 0, 1, 2, \ldots)$ are the coefficients of the Fourier functions $\tilde{u}_1^{(i+1)}(x, y, T'^{(i)}) - v(x, y)$ and $\tilde{u}_2^{(i+1)}(x, y, T_1'^{(i+1)}) - w(x, y)$, respectively; $T_1^{(i+1)}$ and $T^{(i+1)}$ are smallest roots of the

$$\tilde{u}_1^{(i+1)}(x^0, y^0, T_1'^{(i+1)}) = u_*, \; \tilde{u}_2^{(i+1)}(x^0, y^0, T'^{(i+1)}) = u_{**}. \tag{2.5}$$

Formulas (2.3)-(2.5) are valid for the case $\psi(x^0, y^0) < u_*$.

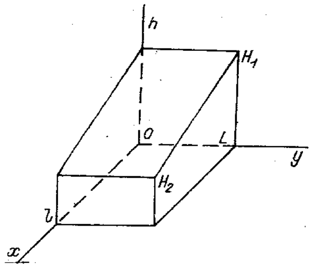

Fig. 1

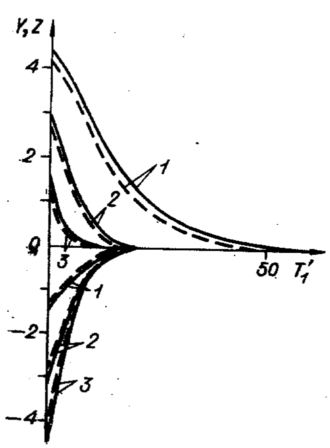

Fig. 2

It is shown in [2] that the problem, similar to Eqs. (1.1)-(1.3) and (2.1), for the case when the groundwater occupies the region between two parallel channels or drains, is characterized by four cases. It can be shown that the problem (1.1)-(1.3), (2.1) also is characterized by four cases:

$$u_{**} < w(x^0, y^0) < v(x^0, y^0) < u_*;$$
$$w(x^0, y^0) < u_{**} < v(x^0, y^0) < u_*;$$
$$u_{**} < w(x^0, y^0) < u_* < v(x^0, y^0);$$
$$w(x^0, y^0) < u_{**} < u_* < v(x^0, y^0).$$

In accordance with the results obtained in [2], in the first three cases there is a value of s such that $T_1^{(i+1)} \underset{i\to s}{\to} \infty$ or $T^{(i+1)} \underset{i\to s}{\to} \infty$. The corresponding formulas (2.5) (first or second) for i+1=s in this case are invalid. For the fourth case, formulas (2.3)-(2.5) are valid for any values of i and give an oscillating solution. Similar formulas are valid also for the case $\psi(x^0, y^0) > u_*$. Thus, in the fourth case, an optimum regime of irrigation is obtained: owing to the fluctuations of the groundwater level between values less than $v_c(x, y)$ and greater than $w_c(x, y)$, where, in accordance with Eqs. (1.4), (1.7), and (1.8), we denote

$$w_c(x, y) = H_1 + (H_2 - H_1)\cdot x/l + w(x, y);\ v_c(x, y) = H_1 + (H_2 - H_1)\cdot x/l + v(x, y),$$

an excessive rise of the groundwater level should be avoided and, consequently, salinization and an excessive lowering of this level, i.e., depletion of the water-bearing stratum. For this, it is sufficient to choose h_* and h_{**} so that the fourth case is valid:

$$w_c(x^0, y^0) < h_{**} < h_* < v_c(x^0, y^0).$$

Let us dwell in more detail on the fourth case, and let us show that with an unlimited increase of time the solution of Eq. (2.3) of the problem tends to the periodic solution of Eqs. (1.8) and (1.9).

It is clear that the equations

$$\tilde{u}_1^{(i+1)}(x, y, T_1'^{(i+1)}) = \tilde{u}_2^{(i+1)}(x, y, T_1'^{(i+1)}),$$
$$\tilde{u}_2^{(i+1)}(x, y, T'^{(i+1)}) = \tilde{u}_1^{(i+2)}(x, y, T'^{(i+1)}) \qquad (2.6)$$

must be satisfied. Introducing the notation

$$\beta_{k,m}^{(i+1)} = \exp\left(-\lambda_{k,m}^2\ T_1^{(i+1)}\right),\ \gamma_{k,m}^{(i+1)} = \exp\left[-\lambda_{k,m}^2\left(T^{(i+1)} - T_1^{(i+1)}\right)\right],$$

we rewrite conditions (2.6) in order to find the constants $C_{k,m}^{(i+2)}$ and $D_{k,m}^{(i+1)}$ in the following way:

$$D_{k,m}^{(i+1)} = \beta_{k,m}^{(i+1)} C_{k,m}^{(i+1)} + \theta_{k,m},$$
$$C_{k,m}^{(i+2)} = \gamma_{k,m}^{(i+1)} D_{k,m}^{(i+1)} - \theta_{k,m} \quad (\theta_{k,m} = v_{k,m} - w_{k,m}). \qquad (2.7)$$

Equations (2.7) have the same form as the analogous equations obtained in [1]; therefore, for the coefficients $C_{k,m}^{(i+1)}$ and $D_{k,m}^{(i+1)}$ similar recurrent formulas are valid, taking account of which Eq. (2.5) assumes the form

$$
\begin{aligned}
\beta_1^{(i+1)} &= \beta_1^{(0)} + \sum_{p=3}^{\infty} A_p \{(\beta_1^{(i+1)})^{\mu_p}[1-(\gamma_1^{(i)})^{\mu_p} + (\gamma_1^{(i)}\beta_1^{(i)})^{\mu_p} + \ldots - \\
&- (\gamma_1^{(i)}\beta_1^{(i)}\gamma_1^{(i-1)}\beta_1^{(i-1)} \ldots \gamma_1^{(1)})^{\mu_p}] + \beta_1^{(i+1)}(\gamma_1^{(i)})^{\mu_p}[1-(\beta_1^{(i)})^{\mu_p} + (\beta_1^{(i)}\gamma_1^{(i-1)})^{\mu_p} + \\
&+ \ldots + (\beta_1^{(i)}\gamma_1^{(i-1)} \ldots \gamma_1^{(1)})^{\mu_p}]\} + \sum_{p=2}^{\infty} [\beta_1^{(i+1)} - (\beta_1^{(i+1)})^{\mu_p}]\ (\gamma_1^{(i)}\beta_1^{(i)}\gamma_1^{(i-1)} \ldots \gamma_1^{(1)}\beta_1^{(1)})^{\mu_p}\tilde{C}_p^{(1)}, \\
\gamma_1^{(i+1)} &= \gamma_1^{(0)} + \sum_{p=3}^{\infty} B_p \{(\gamma_1^{(i+1)})^{\mu_p}[1-(\beta_1^{(i+1)})^{\mu_p} + (\beta_1^{(i+1)}\gamma_1^{(i)})^{\mu_p} + \\
&+ \ldots + (\beta_1^{(i+1)}\gamma_1^{(i)} \ldots \gamma_1^{(1)})^{\mu_p}] + \gamma_1^{(i+1)}(\beta_1^{(i+1)})^{\mu_p}[1-(\gamma_1^{(i)})^{\mu_p} + \\
&+ (\gamma_1^{(i)}\beta_1^{(i)})^{\mu_p} + \ldots - (\gamma_1^{(i)}\beta_1^{(i)}\gamma_1^{(i-1)} \ldots \gamma_1^{(1)})^{\mu_p}]\} + \sum_{p=2}^{\infty} [\gamma_1^{(i+1)} - (\gamma_1^{(i+1)})^{\mu_p}](\beta_1^{(i+1)}\gamma_1^{(i)} \ldots \beta_1^{(1)})^{\mu_p}\tilde{\tilde{C}}_p^{(1)}.
\end{aligned} \tag{2.8}
$$

Here we denote

$$
\begin{gathered}
A = u_{**} - w(x^0, y^0) - \theta_{1,1}\sin\frac{\pi x^0}{l}\sin\frac{\pi y^0}{L}, \\
B = u_* - v(x^0, y^0) + \theta_{1,1}\sin\frac{\pi x^0}{l}\sin\frac{\pi y^0}{L}, \\
\beta_1^{(0)} = [u_* - v(x^0, y^0)]/A,\ \gamma_1^{(0)} = [u_{**} - w(x^0, y^0)]/B, \\
A_{k,m} = \theta_{k,m}\sin\frac{\pi k x^0}{l}\sin\frac{\pi m y^0}{L}\Big/A, \\
B_{k,m} = -\theta_{k,m}\sin\frac{\pi k x^0}{l}\sin\frac{\pi m y^0}{L}\Big/B, \\
E_{k,m}^{(1)} = C_{k,m}^{(1)}\sin\frac{\pi k x^0}{l}\sin\frac{\pi m y^0}{L}, \\
\tilde{C}_{k,m}^{(1)} = E_{k,m}^{(1)}/A,\ \tilde{\tilde{C}}_{k,m} = E_{k,m}^{(1)}/B, \\
\beta_p^{(i+1)} = \beta_{k,m}^{(i+1)},\ \gamma_p^{(i+1)} = \gamma_{k,m}^{(i+1)},\ A_p = A_{k,m}, \\
\tilde{C}_p^{(1)} = \tilde{C}_{k,m}^{(1)},\ B_p = B_{k,m},\ \tilde{\tilde{C}}_p^{(1)} = \tilde{\tilde{C}}_{k,m}^{(1)}\ \ (p = 1, 2\ \ldots), \\
\beta_{k,m}^{(i+1)} = [\beta_{1,1}^{(i+1)}]^{\mu_{k,m}},\ \gamma_{k,m}^{(i+1)} = [\gamma_{1,1}^{(i+1)}]^{\mu_{k,m}}, \\
\mu_{k,m} = \left(\frac{\lambda_{k,m}}{\lambda_{1,1}}\right)^2 (k,\ m = 1, 2\ldots),\ \mu_p = \mu_{k,m}\ (p = 1, 2\ldots).
\end{gathered}
$$

For this, it is assumed that $A \neq 0$ and $B \neq 0$.

Equations (2.8) coincide with Eqs. (3.1) and (3.2) of [1] for the plane case. Because of this, all further reasoning given in [1] concerning estimates of the quantities $\beta_{\min}$, $\gamma_{\min}$, $\gamma_{\max}$, and $\beta_{\max}$, which can be found by the method of successive approximations, and concerning the estimates of the average values of the arbitrary functions entering into the right-hand side of Eq. (2.8), remains valid. In this case, in the first of formulas (3.4) of [1], $q = (\beta_{\max}\gamma_{\max})^{\mu_3}$, it is necessary to put $\mu_3 = \min(\mu_{1,3}, \mu_{3,1})$.

In a similar way, the result also is obtained concerning the trend as $i \to \infty$ of the quantities $\beta_1^{(i+1)}$ and $\gamma_1^{(i+1)}$ to the limits β_1 and γ_1, where β_1 and γ_1 are the quantities considered above – $\beta_1 = \exp(-\lambda_{1,1}^2 T_1)$ and $\gamma_1 = \exp[-\lambda_{1,1}^2(T - T_1)]$ – for the periodic solution of Eqs. (1.8) and (1.9).

The cases when one of the two equations $A = 0$ and $B = 0$, or both at once, is satisfied are considered similarly.

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